Malaysian Journal of Mathematical Sciences 16(4): 783–791 (2022)
https://doi.org/10.47836/mjms.16.4.10

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Subset Perfect Codes of Finite Commutative Rings Over Induced Subgraphs of Unit Graphs

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Received: 6 June 2021
Accepted: 6 October 2022

Abstract

The induced subgraph of a unit graph with vertex set as the non unit elements of a ring R is a graph obtained by deleting all unit elements of R. In a graph Γ, a subset of the vertex set is called a perfect code if the balls with radius 1 centred on the subset are pairwise disjoint and their unions yield the whole vertex set. In this paper, we determine the perfect codes of induced subgraphs of the unit graphs associated with some finite commutative rings R with unity that has a vertex set as non unit elements of R. Moreover, we classify the commutative rings in which their associated induced subgraphs of unit graphs admit the trivial and non-trivial perfect codes. We also characterize the commutative rings based on the induced subgraph of unit graphs that do not admit the perfect codes. Furthermore, we prove that the complement induced subgraph of unit graph admit only the trivial subring perfect code.

Keywords: Commutative ring; unit graph; induced subgraph; perfect code.

1 Introduction

A graph is a mathematical model which is formally represented as $\Gamma = (V, E)$, where V and E denote the vertex and edge sets, respectively. Two vertices x and y in a graph Γ is called adjacent if xy forms an edge of the graph. If no such vertex y is adjacent with x, then the vertex x is called isolated. The graph Γ is called trivial if and only if $|V| = 1$. It is called empty if and only if $E = \emptyset$. A graph S is called a subgraph of a graph Γ, written $S \subseteq \Gamma$, if $V(S) \subseteq V(\Gamma)$ and $E(S) \subseteq E(\Gamma)$. A subgraph of Γ is called induced if it is obtained only by vertex deletions.

An element x of a ring R is called a unit if we can find $y \in R$ satisfy $x \cdot y = e$, where e is the identity. The characteristics of a ring R is said to be the smallest positive integer n that satisfies $na = 0,\ \forall a \in R$; otherwise the characteristics of R is defined to be 0. [1].

The first research on investigating the connection between the graph theory and commutative ring theory was conducted by Beck [5] to present the colouring of the commutative ring by developing a zero divisor graph. The vertices of this graph is the ring R and the vertex x which is different from the vertex y are connected if and only if they are zero divisors. Later on, the definition of zero divisor graph has been modified by Anderson and Livingston [2] to develop a better structure for the zero divisors of commutative rings. In [3], the authors concentrated on determining the diameter and girth of the zero divisor graphs and characterized the commutative rings in such away that the diameter of the zero divisor graph is $diam(\Gamma(R)) \leq 2$ or its girth is $gr(\Gamma(R)) \geq 4$. However, Sinha and Kuar [19] proved several results to present the girth property of the zero devisor graph introduced in [5]. In 2016, Binnis *et al.* [6] generalized the zero divisor graph by introducing a new notion of graph called extension zero divisor graph. In addition, they investigated the diameter (*diam*) and girth (*gr*) of this graph as well as provided some proofs in which the zero divisor and extended zero divisor graphs coincide. In [17], the definition of a zero divisor graph was extended to a non-commutative ring. The authors defined the directed and undirected zero divisor graph for a non-commutative rings and presented their properties, namely diameter and girth.

The notion of unit graph of a ring R was defined in [4]; it is a graph with vertices as elements of R and distinct vertices x and y are connected if and only if $x + y$ is a unit element of R. Besides, in [4], some conditions have been provided in which the unit graph of rings to be cycle, complete and complete bi-partite graphs. It is obvious that all unit graphs are not connected, therefore some results have been established which show the connectedness of the unit graphs. The diameter and girth properties of the unit graph have also been investigated in [4], it was proven that $diam(\Gamma(R))$ is either $1, 2, 3$ or ∞ and $gr(\Gamma(R))$ is either $3, 4, 6$ or ∞. Su *et al.* [21] concentrated on the planarity property of the unit graph and provided some necessary and sufficient conditions for the unit graphs to be planar. In [12], the finite commutative rings with unity have been classified based on their associated unit graphs having dominating number less than 4. Other research articles are also devoted to the unit graph of ring R (see, [20], [23], [22], [16]).

The theory of coding was introduced in 20th century in engineering to solve the problem arose concerning the efficient transmitting and storing information. This study has been originated in the landmark papers authored by Shannon [18] and Hamming [10]. In the classical setting, a code C is defined to be a subset of the set of all words F_q^n over the set of alphabets. It is called perfect code if every element in the set F_q^n is of distance not more than 1 to exactly one element of C. In graph point of view, a perfect code can be thought as an efficient dominating set [9] or independent perfect dominating set [13] of the graph. In view of foregoing, any non empty subset C of the vertex set $V(\Gamma(R))$ is called a code. The code C is a perfect code in Γ if $S_1(c)$, as c runs through C, forms a partition of $V(\Gamma(R))$ [7]. In other words, the code C is a perfect code in Γ if

$\bigcup S_1(x) = V(\Gamma(R))$ for all $x \in C$ and $S_1(x) \cap S_1(y) = \emptyset$ for each distinct elements $x, y \in C$, where $S_1(x)$ is the closed neighbourhood of x and $S_1(y)$ is the closed neighbourhood of y [24]. The first investigation on studying perfect codes in graph was conducted in [7] to establish a connection between coding theory and graph theory. Later on, this investigation was extended to Cayley graph of a group in [11]. Chen *et al.* [8] also continued the research on perfect codes in Cayley graph of finite groups and provided some necessary and sufficient conditions in which the Cayley graph admits the subgroup of a finite group G as a perfect code. Ma [14] proved some necessary and sufficient conditions in which the propoer reduced power graphs associated with finite groups admit the perfect codes and total perfect codes; particularly he characterized all finite groups in which their associated proper reduced power graph admit an order 2 total perfect code. In 2021, Zhang and Zhou [25] focused on some families of groups namely, 2-groups, metabelian groups, nilpotent groups and generalized dihedral groups associated with Cayley graphs and provided some conditions for their subgroups to be perfect codes. Recently, Ma *et al.* [15] focused on the perfect codes of Cayley sum graphs of finite abelian groups G and established some necessary and sufficient conditions for the Cayley sum graphs admitting a subset of G as a perfect code.

In this paper, the study of perfect codes in graphs associated with groups is extended in graph associated with rings. Particularly, we determine the trivial and non-trivial subsets of commutative rings R with unity to be perfect codes in induced subgraphs of unit graphs with vertex set non-unit elements of R. In Section 2, we classify commutative ring with identity in which their associated induced subgraph of unit graphs admit the trivial and non-trivial perfect codes. We also characterize the commutative rings in which their associated induced subgraphs of unit graphs do not admit the perfect codes. In Section 3, we prove that the complement induced subgraph of unit graph admit only the trivial subring perfect code.

Other notations and terminologies used in this paper are R shows the finite commutative ring with identity and $O(R)$, $Char(R)$, $U(R)$ and $NU(R)$ denote the order, characteristics, sets of units and non units of R, respectively. The unit graph associated with R is a simple, undirected and finite graph and is denoted by $\Gamma(R)$. In addition, $\Gamma_1(R)$ and $\Gamma_1^c(R)$ denote the induced subgraph and complement induced subgraph of the unit graph and C represents the perfect codes in these graphs.

2 Perfect Codes in Induced Subgraph of a Unit Graph

In this section, some necessary and sufficient conditions are established to characterize the commutative rings in which their associated induced subgraphs of unit graphs admit the trivial and non-trivial perfect codes. A perfect code C in $\Gamma_1(R)$ is a trivial perfect code if $|C| = 1$ or $C = V(\Gamma_1(R))$, otherwise C is a non-trivial perfect code. Therefore, to characterize R with respect to the induced subgraph of a unit graph, two necessary and sufficient conditions are provided on determining the trivial perfect code of order 1 (Proposition 2.1) and a trivial perfect code of order cardinality of the vertex set (Theorem 2.1). Furthermore, two necessary and sufficient conditions are established to show a perfect code of order 2 (Theorem 2.2) and a perfect code of order 2^{n-1}, where $n \geq 2$ (Theorem 2.3) over induced subgraphs of the unit graphs of R. Additional condition is also provided to characterize the commutative rings in which their associated induced subgraphs of unit graphs do not admit the perfect codes (Theorem 2.4).

Proposition 2.1. *If $\Gamma_1(R)$ is the induced subgraph of the unit graph associated with a ring R, then the statements that follows are equivalent:*
(i) *$O(R) = p$, p is a prime;*
(ii) *$\Gamma_1(R) = K_1$, where K_1 is a complete graph of order 1;*

(iii) $|C| = 1$.

Proof. (i) $\to$ (ii) Assume R is a ring of $O(R) = p$, where p is prime. This gives that $|NU(R)| = 1$. Since $V(\Gamma_1(R)) = NU(R)$, this implies that $\Gamma_1(R) = K_1$, which is a trivial graph.

(ii) $\to$ (iii) Since $\Gamma_1(R)$ is a trivial graph, this gives $C = \{0\}$ is the trivial subring perfect code. Hence, $|C| = 1$.

(iii) $\to$ (i) If $|C| = 1$, it means that the set $\{0\}$ is the set of non unit of the ring R. This implies $O(R) = p$. □

Theorem 2.1. *If $\Gamma_1(R)$ is the induced subgraph of the unit graph associated with a ring R, then the statements that follows are equivalent:*
(i) $Char(R) = O(R) = p^k$*, where* p *is prime and* $k \geq 2$*;*
(ii) $\Gamma_1(R) = \overline{K}_m$*, where* $m = |NU(R)|$*;*
(iii) $|C| = m$.

Proof. (i) $\to$ (ii) Assume R is a ring of $Char(R) = O(R) = p^k$, where p is prime and $k \geq 2$. Then $|NU(R)| \geq 2$, since all elements of multiples of p are not units in R. Let r and s be two different elements in $NU(R)$, thus $r + s \in NU(R)$, that is the vertices r and s are not adjacent. This yields that $\Gamma_1(R)$ is an empty graph with $m = |NU(R)|$ vertices, that is $\Gamma_1(R) = \overline{K}_m$.

(ii) $\to$ (iii) Let $\Gamma_1(R) = \overline{K}_m$, where $m = |NU(R)|$, then $C = NU(R) = \{x_1, x_2, ..., x_m\}$ is the perfect code, since $S_1(x_k) \cap S_1(x_l) = \emptyset$ for all $x_k \neq x_l,\ x_k, x_l \in C$ and $\bigcup_{k=1}^{m} S_1(x_k) = V(\Gamma_1(R))$. Hence, $|C| = m$.

(iii) $\to$ (i) If $|C| = m$, where $m = |NU(R)|$, this means that for $x_k \in C$, $S_1(x_k)$ partitions the vertex set $V(\Gamma_1(R))$ into m singleton sets. It follows that there is no connection between any two distinct elements in $V(\Gamma_1(R))$. This implies that all elements of $V(\Gamma_1(R))$ are multiples of p. Hence, $Char(R) = O(R) = p^k$. □

Theorem 2.2. *If $\Gamma_1(R)$ is the induced subgraph of the unit graph associated with a ring R, then the statements that follows are equivalent:*
(i) $Char(R) = O(R) = 2p$*, where* p *is an odd prime;*
(ii) $\Gamma_1(R) = K_1 + K_{1,n}$*, where* $n = |NU(R)| - 2$*;*
(iii) $|C| = 2$.

Proof. (i) $\to$ (ii) Assume R is a ring of $Char(R) = O(R) = 2p$, where p is an odd prime, then $|NU(R)| > 3$, since all elements of multiples of 2 and p are non units. Let $x \in NU(R)$ and $x \neq 0$, then $0 + x \in NU(R)$. Thus, 0 is an isolated vertex in $\Gamma_1(R)$. Similarly, there exists an element $y \neq 0$ in $NU(R)$ (y is the only non zero element of multiples of p in R) such that $y + x \in U(R)$. This gives that $\Gamma_1(R) = K_1 + K_{1,n}$, where $n = |NU(R)| - 2$.

(ii) $\to$ (iii) If $\Gamma_1(R) = K_1 + K_{1,n}$, then $C = \{0, y\}$ is the perfect code in $\Gamma_1(R)$, since $S_1(0) \cap S_1(y) = \emptyset$ and $S_1(0) \cup S_1(y) = V(\Gamma_1(R))$. Hence, $|C| = 2$.

(iii) $\to$ (i) Suppose $|C| = 2$, that is when r_k runs through C, $S_1(r_k)$ partitions the vertex set $V(\Gamma_1(R))$ into two distinct sets. Since, $0 \in V(\Gamma_1(R))$, thus the partition sets of $V(\Gamma_1(R))$ are $S_1(0) = \{0\}$ and $S_1(y) = \{x \neq 0 : x \in NU(R)\}$. This shows that the set $S_1(y)$ contains an element $y \neq 0$ such that $y + x \in U(R)$. Thus, y is the only non zero element of multiples of p, $p \geq 3$ in R which is adjacent to every non zero element of multiples of 2 in R. Therefore, $Char(R) =$

$O(R) = 2p$. Let $Char(R) = O(R) \neq 2p$, then there exists no such element $y \in NU(R)$ such that $S_1(y) = \{x \neq 0 : x \in NU(R)\}$, which is a contradiction. Hence, the theorem is proved. $\square$

In the following, an example is provided to illustrate Theorem 2.2.

Example 2.1. *Let $R = \mathbb{Z}_{62}$, where $Char(R) = O(R) = 2 \cdot 31$. By Theorem 2.2, $\Gamma_1(R) = K_1 + K_{1,30}$, where $K_{1,30}$ is a star graph as shown in Figure 1.*

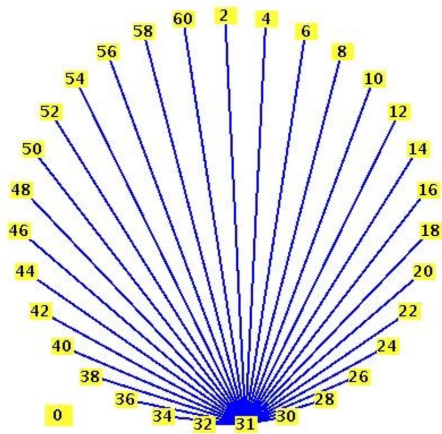

Figure 1: The induced subgraph of unit graph of the ring $R = \mathbb{Z}_{62}$.

According to Theorem 2.2, $C = \{0, 31\}$ is the perfect code in $\Gamma_1(R)$. In the next theorem, the rings R are characterized in which their associated induced subgraphs of the unit graphs admit a perfect code of order 2^{n-1}, where $n \geq 2$.

Theorem 2.3. *If $\Gamma_1(R)$ is the induced subgraph of the unit graph associated with a ring R, then the statements that follows are equivalent:*
(i) $Char(R) = 2$ and $O(R) = 2^n$, where $n \geq 2$;
(ii) $\Gamma_1(R) = K_1 + \bigcup_{k=1}^{m} K_2$, where $m = 2^{n-1} - 1$;
(iii) $|C| = 2^{n-1}$.

Proof. (i) $\to$ (ii) Assume R is a ring of $Char(R) = 2$ and $O(R) = 2^n$, where $n \geq 2$, then the only unit of R is e. Since, R has 2^n elements, thus $|NU(R)| = 2^n - 1$. Suppose that $x \neq 0$, $x \in NU(R)$, then $0 + x \in NU(R)$. Therefore 0 is an isolated vertex. Similarly, by the vertices adjacency of $\Gamma_1(R)$, it can be seen that there are $m = 2^{n-1} - 1$ distinct pairs of $x, y \in NU(R)$ which satisfy the condition $x + y \in U(R)$. That is each distinct pair of x and y forms a K_2. Therefore, $\Gamma_1(R)$ contains an isolated vertex with $2^{n-1} - 1$ components of K_2, i.e. $\Gamma_1(R) = K_1 + \bigcup_{k=1}^{m} K_2$, where $m = 2^{n-1} - 1$.

(ii) $\to$ (iii) Suppose that $\Gamma_1(R) = K_1 + \bigcup_{k=1}^{m} K_2$, where $m = 2^{n-1} - 1$, then we need to show that $|C| = 2^{n-1}$. Let $C \subseteq V(\Gamma_1(R))$ be a code, then $C = \{c_i : i = 1, 2, ..., 2^{n-1}\}$, where c_i is not adjacent to c_j for $i \neq j$ is a perfect code in $\Gamma_1(R)$, since $S_1(c_i) \cap S_1(c_j) = \emptyset$ for all distinct $c_i, c_j \in C$ and $\bigcup_{i=1}^{2^{n-1}} S_1(c_i) = V(\Gamma_1(R))$. Hence, $|C| = 2^{n-1}$.

(iii) $\to$ (i) Assume $|C| = 2^{n-1}$, that is when c_i runs through C, $S_1(c_i)$ partition the vertex set $V(\Gamma_1(R))$ into 2^{n-1} distinct sets of order $|S_1(c_1 = 0)| = 1$ and $|S_1(c_i)| = 2$ for $i = 2, 3, 4, ..., 2^{n-1}$.

This gives that $|NU(R)| = 2^n - 1$, $n \geq 2$ and $|U(R)| = 1$. Hence, $Char(R) = 2$ and $O(R) = 2^n$, where $n \geq 2$. $\square$

Example 2.2. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with $Char(R) = 2$ and $O(R) = 2^3$. By Theorem 2.3, $\Gamma_1(R) = K_1 + 3K_2$ as shown in Figure 2.*

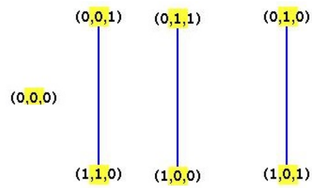

Figure 2: The induced subgraph of unit graph of the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

By Theorem 2.3, $C = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1)\}$ is a perfect code in $\Gamma_1(R)$.

There are commutative rings with identity in which their associated induced subgraphs of unit graphs do not admit the perfect codes. This result is presented in theorem 2.4.

Theorem 2.4. *If R is a ring with $Char(R) = O(R) = pq$, where p and q are both odd primes $p \neq q$ and $\Gamma_1(R)$ be the induced subgraph of the unit graph, then the following hold true:*
(i) $\Gamma_1(R) = K_1 + K_{m,n}$, where $K_{m,n}$ is a complete bipartite graph;
(ii) $\Gamma_1(R)$ does not admit the perfect code.

Proof. (i) Assume R is a ring of $Char(R) = O(R) = pq$, where p and q are both odd primes $p \neq q$, then $NU(R)$ contains all elements of multiples of p and multiples of q. Let $x_i \neq 0$ indicate the elements of multiples of p and $y_i \neq 0$ indicate the elements of multiples of q, then $x_i + y_i \in U(R)$, while $x_i + x_j \in NU(R)$ and $y_i + y_j \in NU(R)$ for $i \neq j$. This shows that the non zero non unit elements form a $K_{m,n}$ graph with partite sets containing $m > 1$ and $n > 1$ elements, respectively. Similarly, since $0 \in NU(R)$ and $0 + x_i \in NU(R)$ and $0 + y_i \in NU(R)$ for any $x_i, y_i \in NU(R)$, therefore 0 is an isolated vertex. Hence, $\Gamma_1(R) = K_1 + K_{m,n}$.

(ii) Let $\Gamma_1(R) = K_1 + K_{m,n}$, then it is shown that $\Gamma_1(R)$ does not admit the perfect code. Let $C_1 \subseteq V(K_{m,n})$ be a code, to show C_1 is not a perfect code admitted by $K_{m,n}$, two cases are considered.

Case 1: If $|C_1| = 1$, then $S_1(c) \neq V(K_{m,n})$ for $c \in C_1$. Hence, $K_{m,n}$ does not admit the perfect code of order 1.

If $|C_1| > 1$, then $S_1(c_i) \cap S_1(c_j) \neq \emptyset$ for all distinct $c_i, c_j \in C_1$, which does not satisfy the condition of a perfect code. Hence, $K_{m,n}$ does not admit the perfect code of order greater than 1.

Consequently, $\Gamma_1(R)$ does not admit the perfect code. $\square$

Example 2.3. *Let $R = \mathbb{Z}_{39}$, where $Char(R) = O(R) = 3 \cdot 13$. By Theorem 2.4, $\Gamma_1(R) = K_1 + K_{2,12}$ as shown in Figure 3.*

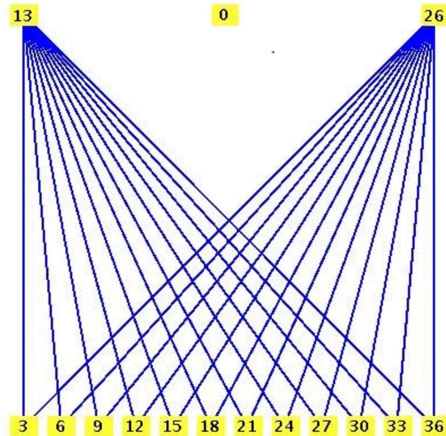

Figure 3: The induced subgraph of unit graph of the ring $R = \mathbb{Z}_{39}$.

By Theorem 2.4, $\Gamma_1(R)$ does not admit the perfect code.

However, the complement induced subgraph of a unit graph with vertex set non-unit elements of ring R admits only the trivial perfect code. This results is proved in Theorem 2.5.

Theorem 2.5. *If $\Gamma_1^c(R)$ is the complement induced subgraph of the unit graph associated with a ring R, then C is a trivial subring perfect code and $|C| = 1$.*

Proof. Assume R is a commutative ring with unity. Since $0 \in NU(R)$, then $\Gamma_1(R)$ contains K_1 as its components. It follows that $\Gamma_1^c(R)$ contains a vertex 0 of $deg(0) = |NU(R)| - 1$, that is $0 + s \in NU(R)$, where $s \neq 0$ is an element in $NU(R)$. Let $C \subseteq V(\Gamma_1^c(R))$ be a code, then $C = \{0\}$ is a trivial subring perfect code in $\Gamma_1^c(R)$, since $S_1(0) = V(\Gamma_1^c(R))$. Hence, $|C| = 1$. □

3 Conclusions

This paper provides some necessary and sufficient conditions on determining the perfect codes of induced subgraph of a unit graph associated with some finite commutative rings R. The findings show that if R has order p, then the trivial subring is the perfect code, while if $Char(R) = O(R) = p^k, k \geq 2$, then the set of non-unit elements is the perfect code. Moreover, if $Char(R) = O(R) = 2p, p \geq 3$, then an order 2 subset of ring R is the perfect code, while if $Char(R) = 2$ and $O(R) = 2^n,\ n \geq 2$, then an order 2^{n-1} subset of the ring R is the perfect code. However, if $Char(R) = O(R) = pq,\ p \neq q$, p and q are both odd primes, then no perfect code exists. The findings also show that the complement induced subgraph of a unit graph admit only the trivial subring perfect code.

Acknowledgement The authors would like to acknowledge Universiti Teknologi Malaysia (UTM) and Research Management Centre (RMC) UTM for the financial funding through the UTM Fundamental Research Grant Vote No. 20H70 and Fundamental Research Grant Scheme (FRGS1/2020/STG06/UTM/01/2). The first author would also like to thank the Ministry of Higher Education (MOHE) of Afghanistan, particularly Higher Education Development Program (HEDP) for his scholarship and Kabul University for the study leave. The third author would also like to thank UTM for his post doctoral fellowship.

Conflicts of Interest The authors declare that we have no conflicts of interest to reveal.

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